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H_∞ and H_2 optimizations of a dynamic vibration absorber for suppressing vibrations in plates

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Abstract

H_∞ and H_2 optimization problems with respect to a dynamic vibration absorber (DVA) in a single degree-of-freedom (sdof) system are classical optimization problems and solutions to them were found about half a century ago. Numerical solutions to the H_∞ and H_2 optimization problems with respect to DVA for a multi-degree-of-freedom (mdof) or continuous system can be found in the literature but their analytical solutions have not yet been found. In this article, we report the derivation of an analytical solution to the H_∞ and H_2 optimization problems of DVA applied to suppress random vibrations in plate structures. Analytical formulae are also proposed to express the optimal tuning frequency and damping ratios of the absorber. The established theory improves our understanding of the effects of different parameters including the mass, damping and tuning ratios and also the point of attachment of the absorber on the vibration absorption by the absorber. Numerical results show the usefulness of the optimization solutions in comparison to solutions suggested by other researchers based on other approaches to the problem.

1. Introduction

Using a passively tuned dynamic vibration absorber (DVA) is one way to suppress random vibration in mechanical and civil structures. It was invented by Frahm [1] in 1911. In 1928, Ormondroyd and Den Hartog [2] pointed out that the damping of the DVA had an optimum value for the minimization of the amplitude response of the sdof system at resonance. Such optimization criterion is now known as H_∞ optimization. The fixed-points theory of Den Hartog [3] was commonly used for the determination of the optimum tuning frequency and damping ratios of the DVA attached to an sdof vibrating system.

H_2 optimization of the vibration absorber has the objective function of minimizing the total vibration energy of the primary structure under white noise excitation. In 1963, Crandall and Mark [4] found out the H_2 optimized tuning frequency and damping ratios for the sdof system.

Some recent research work on the optimum tuning of DVA for sdof systems can be found in the reports of Asami et al. [5,6]. They derived the analytical solutions to H_∞ and H_2 optimization problems of DVAs

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Nomenclature

a	Fourier coefficient of g
b	Fourier coefficient of δ
c	damping coefficient of the absorber
D	flexural rigidity
E	modulus of elasticity
f	normalized frequency
g	deterministic spatial function of the externally applied force
h	thickness of the plate
j	$=\sqrt{-1}$
k	stiffness of the absorber
L_x, L_y	length and width of the plate
m	mass of the absorber
n	Fourier coefficient of u
N	Laplace transform of n
p, q	indices of the eigenmodes
R	Laplace transform of r
s	Laplace variable
S	spectral density of the vibration response of the plate
t	time variable
T	tuning ratio of the absorber
u	dynamic displacement of the plate
U	Laplace transform of u
$\dot{U}$	velocity amplitude of the plate
$\ddot{U}$	acceleration amplitude of the plate
w	stationary random function of time of the externally applied force
W	Laplace transform of w
x, y	spatial variables
Z	non-dimensional frequency response function of the plate
v	Poisson ratio
ρ	material density
ζ	damping ratio of the absorber
φ	eigenfunction of the plate without the absorber
γ	non-dimensional natural frequency of the plate
ω	natural frequency of the plate without the absorber
ω_a	undamped natural frequency of the absorber
$\omega_{\alpha\beta}$, $\varphi_{\alpha\beta}$	frequency and shape of the $\alpha\beta$th mode at which the absorber is tuned for vibration control
μ	mass ratio between the masses of the absorber and the plate
ε	equivalent mass ratio

attached to damped linear systems. Research works found in the literature on the optimum tuning of DVA for the mdof system or the continuous system are mostly numerical optimization methods. Rice [7] reported the use of a SIMPLEX nonlinear optimization procedure to determine the H_∞ optimum tuning of a vibration absorber applied for suppressing the vibration of a beam. Hadi and Arfiadi [8] used a genetic algorithm to solve numerically the H_2-optimized tuning frequency and damping ratios for mdof systems. Jacquot [9] proposed a transfer function of the plate attached with a DVA. He set the frequency ratio to one and determined the optimum damping ratio of the absorber based on the transfer function. However, it is shown in the latter sections of this article that the optimum frequency ratio of the absorber is in general not equal to one and another set of optimum frequency and damping ratios has been derived based on the proposed analytical model. Dayou [10] applied the fixed-points theory [3] and proposed a set of optimum frequency and damping ratios for global control of the kinetic energy of a continuous structure using DVA. Zuo and Nayfeh [11] and Wong et al. [12] studied DVA of two-degree-of-freedom for suppressing vibration in the sdof system and in beams under forced single harmonic vibration, respectively.

In this article, a theory is established for describing the excitation–response relation leading to the H_∞ and H_2 optimum tuning of the DVA attached onto a plate structure. The present case is much more complicated than an sdof structure because an improper selection of attachment point for the absorber may lead to an amplification of vibration in other parts of the structure [12]. The established theory improves our understanding of the effects of different parameters including the mass, damping and tuning ratios and also the point of attachment of the absorber on the vibration absorption by the absorber. The optimum tuning as derived in this article based on the fixed-points theory [3] includes tuning frequency and damping ratios of the absorber and also the position of the absorber on the vibrating structure. The objective of the optimum tuning is to minimize vibrational displacement, velocity and acceleration of a point on the plate as well as the minimization of root mean square motion over the whole domain of the plate. The numerical simulations are used to show the usefulness of the optimization solutions leading to better vibration control in continuous systems than those suggested by other researchers [9,10] based on other approaches to the problem.

2. Theory

Consider a thin rectangular plate on the rectangular domain $0 \leqslant x \leqslant L_x$ and $0 \leqslant y \leqslant L_y$ that carries a DVA at point (x_o, y_o) as shown in Fig. 1. The plate is under external distributed force $w(t)g(x,y)$ and the point force $r(t)$ is transmitted to the plate by the attached dynamic absorber. The equation of motion for the plate may be written as

$$\nabla^4 u + \frac{\rho h}{D}\frac{\partial^2 u}{\partial t^2} = \frac{w(t)g(x,y)}{D} + \frac{r(t)}{D}\delta(x - x_o)\delta(y - y_o), \tag{1}$$

where the flexural rigidity of the plate D is defined as

$$D = \frac{Eh^3}{12(1 - v^2)} \tag{2}$$

and where E is the modulus of elasticity, v the Poisson ratio, h the thickness of the plate and ρ the material density.

It is assumed that the externally applied forcing function is $w(t)g(x,y)$, where $g(x,y)$ is a deterministic function of x and y, and $w(t)$ is a stationary random function of time. The equation of motion of the free vibration of the plate without the absorber may be written as

$$\nabla^4 \varphi_{pq}(x,y) = \frac{\rho h}{D}\omega_{pq}^2 \varphi_{pq}(x,y), \tag{3}$$

where ω_{pq} and $\varphi_{pq}(x,y)$ are the pqth natural frequency and eigenfunction of the plate without the absorber, respectively. The solution to Eq. (1) can be expanded in a Fourier series written as

$$u(x,y,t) = \sum_{p=1,q=1}^{\infty} n_{pq}(t)\varphi_{pq}(x,y). \tag{4}$$

Similarly, the spatial part of the forcing function can be expanded as

$$g(x,y) = \sum_{p=1,q=1}^{\infty} a_{pq}\varphi_{pq}(x,y). \tag{5}$$

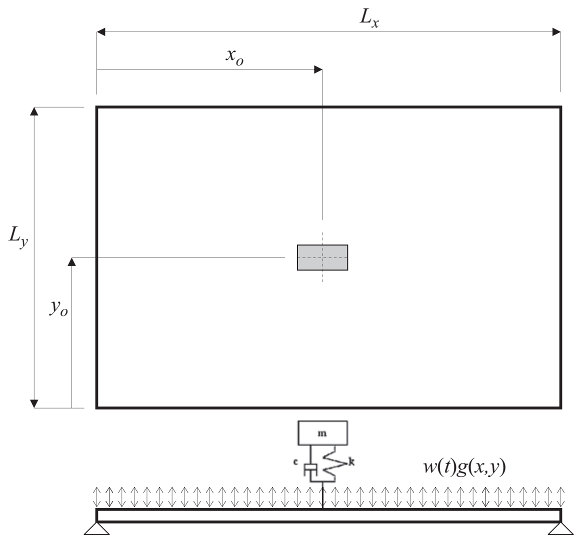

Fig. 1. A simply supported rectangular plate under external distributed force f and carrying a dynamic vibration absorber at point (x_o, y_o).

The Dirac delta functions can also be expanded as

$$\delta(x - x_o)\delta(y - y_o) = \sum_{p=1,q=1}^{\infty} b_{pq}\varphi_{pq}(x, y), \tag{6}$$

where the Fourier coefficients a_{pq} and b_{pq} are, respectively,

$$a_{pq} = \left(\frac{1}{L_x L_y}\right) \int_0^a \int_0^b g(x, y)\varphi_{pq}(x, y)\,\mathrm{d}y\,\mathrm{d}x \tag{7}$$

and

$$b_{pq} = \left(\frac{1}{L_x L_y}\right) \varphi_{pq}(x_o, y_o). \tag{8}$$

Substituting Eqs. (4)–(6) into Eq. (1) and performing Laplace transformation on the resulting equation with respect to time, the result may be written as

$$\sum_{p=1,q=1}^{\infty} \left[\frac{\rho h}{D}\omega_{pq}^2 N_{pq}(s) + \frac{\rho h}{D}s^2 N_{pq}(s) - \frac{a_{pq}}{D}W(s) - \frac{b_{pq}}{D}R(s)\right]\varphi_{ij}(x, y) = 0, \quad p, q = 1, 2, 3, \ldots, \tag{9}$$

where $N_{pq}(s)$, $R(s)$and $W(s)$ are the Laplace transform of $n_{pq}(t)$, $r(t)$ and $w(t)$, respectively.

Since the eigenvectors $\varphi_{pq}(x)$ are linearly independent, we may write

$$\frac{\rho h}{D}\omega_{pq}^2 N_{pq}(s) + \frac{\rho h}{D}s^2 N_{pq}(s) - \frac{a_{pq}}{D}W(s) - \frac{b_{pq}}{D}R(s) = 0, \quad p, q = 1, 2, 3, \ldots. \tag{10}$$

From Eq. (10) above, the generalized co-ordinates $N_{pq}(s)$ may be written as

$$N_{pq}(s) = \left(\frac{1}{\rho h}\right)\left[\frac{a_{pq}W(s) + b_{pq}R(s)}{\omega_{pq}^2 + s^2}\right]. \tag{11}$$

Performing Laplace transformation on Eq. (4) and eliminating $N_{pq}(s)$ in the resulting equation with Eq. (11), the s-domain motion of any point on the plate could be written as

$$U(x, y, s) = \frac{1}{\rho h}\sum_{p=1,q=1}^{\infty}\left[\frac{a_{pq}W(s) + b_{pq}R(s)}{\omega_{pq}^2 + s^2}\right]\varphi_{pq}(x, y), \tag{12}$$

where $U(x, y, s)$ is the Laplace transform of $u(x, y, t)$ with respect to time.

The force transmitted to the beam at the point of attachment may be written as

$$R(s) = -U(x_o, y_o, s)\left[\frac{ms^2(cs + k)}{ms^2 + cs + k}\right]. \tag{13}$$

The functions $R(s)$ in Eq. (12) can be eliminated using Eq. (13) to give

$$U(x, y, s) = \frac{1}{\rho h}\sum_{p=1,q=1}^{\infty}\left[\frac{a_{pq}W(s) - b_{pq}\dfrac{ms^2(cs + k)}{ms^2 + cs + k}U(x_o, y_o, s)}{\omega_{pq}^2 + s^2}\right]\varphi_{pq}(x, y). \tag{14}$$

This expresses the motion of an arbitrary point (x, y) on the vibrating plate in terms of the forcing function $W(s)$ and the motion at the point of attachment (x_o, y_o). This relation would definitely be valid at the attachment point (x_o, y_o) leading to

$$U(x_o, y_o, s) = \frac{1}{\rho h}\sum_{p=1,q=1}^{\infty}\left[\frac{a_{pq}W(s) - b_{pq}\dfrac{ms^2(cs + k)}{ms^2 + cs + k}U(x_o, y_o, s)}{\omega_{pq}^2 + s^2}\right]\varphi_{pq}(x, y). \tag{15}$$

This can be rearranged to arrive at a transfer function between $W(s)$ and $U(x_o, y_o, s)$ written as

$$\frac{U(x_o, y_o, s)}{W(s)} = \frac{\frac{1}{\rho h}\sum_{p=1,q=1}^{\infty}\frac{a_{pq}\varphi_{pq}(x_o, y_o)}{\omega_{pq}^2 + s^2}}{1 + \frac{1}{\rho h}\left[\frac{ms^2(cs+k)}{ms^2+cs+k}\right]\sum_{p=1,q=1}^{\infty}\frac{b_{ij}\varphi_{ij}(x_o, y_o)}{\omega_{pq}^2 + s^2}}. \tag{16}$$

Now it is appropriate to define the following non–dimensional parameters:$\mu = m/\rho h L_x L_y$ is the mass ratio between the masses of the absorber and the plate; $\zeta = c/2\sqrt{mk}$ the damping ratio of the absorber; $\omega_a = \sqrt{k/m}$ the undamped natural frequency of the absorber; $T = \omega_a/\omega_{\alpha\beta}$ is the ratio between the absorber frequency and a reference natural frequency $\omega_{\alpha\beta}$ of the plate; $\gamma_{pq} = \omega_{pq}/\omega_{\alpha\beta}$ is the non-dimensional natural frequency of the plate referred to $\omega_{\alpha\beta}$ and $f = \omega/\omega_{\alpha\beta}$ is the normalized frequency.

The frequency response function of the plate can be obtained by substituting Eq. (16) into Eq. (14) and replacing s by $\mathrm{j}\omega$ in the resulting equation written in the non-dimensional form as

$$\frac{U(x,y,f)}{W(f)} = \frac{1}{\rho h\omega_{\alpha\beta}^2}\sum_{p=1,q=1}^{\infty}\left\{\frac{a_{pq} - b_{pq}\left[\frac{\mu ab\sum_{p=1,q=1}^{\infty}\frac{a_{pq}\varphi_{pq}(x_o, y_o)}{\gamma_{pq}^2 - f^2}}{\frac{-f^2 + 2\zeta Tf + T^2}{-f^2(2\zeta Tf + T^2)} + \mu ab\sum_{i=1,j=1}^{\infty}\frac{b_{pq}\varphi_{pq}(x_o, y_o)}{\gamma_{pq}^2 - f^2}}\right]}{\gamma_{pq}^2 - f^2}\right\}\varphi_{pq}(x,y), \tag{17}$$

where $\mathrm{j} = \sqrt{-1}$. The transfer functions of the velocity and the acceleration responses at point (x, y) on the plate surface may be written, respectively, as

$$\frac{\dot{U}(x,y,f)}{\omega_{\alpha\beta}W(f)} = \mathrm{j}f\frac{U(x,y,f)}{W(f)} \tag{18}$$

and

$$\frac{\ddot{U}(x,y,f)}{\omega_{\alpha\beta}^2 W(f)} = -f^2\frac{U(x,y,f)}{W(f)}. \tag{19}$$

2.1. Optimization for minimizing the vibration at a point (x, y) on the plate

For a structure with well-separated natural frequencies, the modal displacement response in the vicinity of the uvth natural frequency may be approximated by considering $p = \alpha$ and $q = \beta$ and ignoring other modes in Eq. (17). Eq. (17) may then be written as

$$\begin{aligned}\frac{U(x,y,f)}{W(f)} &= \left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\right)\left[\frac{1 - b_{\alpha\beta}\frac{\mu L_x L_y\frac{\varphi_{\alpha\beta}(x_o,y_o)}{1-f^2}}{\left(-\frac{-f^2 + 2\mathrm{j}\zeta Tf + T^2}{f^2(2\mathrm{j}\zeta Tf + T^2)} + \mu L_x L_y\frac{b_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-f^2}\right)}}{1-f^2}\right] \\ &= \left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\right)\left[\frac{T^2 - f^2 + 2\mathrm{j}\zeta Tf}{(2\mathrm{j}\zeta Tf + T^2 - f^2)(1-f^2) - \varepsilon f^2(2\mathrm{j}\zeta Tf + T^2)}\right] = \left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\right)Z(f)\end{aligned} \tag{20}$$

where

$$Z(f) = \frac{T^2 - f^2 + 2\mathrm{j}\zeta Tf}{(2\mathrm{j}\zeta Tf + T^2 - f^2)(1 - f^2) - \varepsilon f^2(2\mathrm{j}\zeta Tf + T^2)} = \left[\frac{\rho h\omega_{\alpha\beta}^2 U(x,y,f)}{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)W(f)}\right] \tag{21a}$$

is the non-dimensional frequency response of the plate, and

$$\varepsilon = \mu\varphi_{\alpha\beta}^2(x_o, y_o). \tag{21b}$$

The objective of H_∞ optimization is to minimize the maximum vibration amplitude response at the point (x, y) and the performance index may be defined as

$$H_{\infty\_\mathrm{pt\_disp}} = \inf_f\left(\sup_{\zeta,T}\left(\left|\frac{U(x,y,f)}{W(f)}\right|\right)\right). \tag{22a}$$

Similarly, the performance indices of H_∞ optimization for minimizing the maximum velocity and acceleration amplitude responses at the point (x, y) may be defined, respectively, as

$$H_{\infty\_\mathrm{pt\_vel}} = \inf_f\left(\sup_{\zeta,T}\left(\left|\frac{\dot{U}(x,y,f)}{\omega_{\alpha\beta}W(f)}\right|\right)\right) \tag{22b}$$

and

$$H_{\infty\_\mathrm{pt\_acc}} = \inf_f\left(\sup_{\zeta,T}\left(\left|\frac{\ddot{U}(x,y,f)}{\omega_{\alpha\beta}^2 W(f)}\right|\right)\right). \tag{22c}$$

The expression $Z(f)$ in Eq. (20) is equivalent to the amplitude ratio as derived by Den Hartog [3] in the sdof system attached with a DVA if the term ε in Eq. (21a) is replaced by the mass ratio μ. ε may therefore be considered as the equivalent mass ratio for applying a vibration absorber to control vibrations in plate structures. The optimum frequency and damping ratios, and the height of the fixed points in the frequency spectrum of the primary system in Eqs. (22a)–(22c) for H_∞ optimization can be derived based on the fixed-points theory in the same way as in the case of the sdof system [5,6], and the results are listed in Table 1.

The objective of H_2 optimization of the absorber is to minimize the total vibration energy of the mass at point (x, y) of the plate of all frequencies in the system and the performance index may be defined as [4]

$$H_{2\_\mathrm{pt}} = \inf_{\zeta,T}\left(\frac{E[U^2]}{2\pi S_F\omega_{\alpha\beta}/D^2}\right), \tag{23}$$

Table 1
The approximated H_∞ tuning of the plate for control of vibration at point (x, y) and the height of the fixed points in the response spectrum

Transfer function	Tuning ratio	Damping ratio	Height of the fixed points in the response spectrum
$\frac{U(x,y,f)}{W(f)}$	$\frac{1}{1+\varepsilon}$	$\sqrt{\frac{3\varepsilon}{8(1+\varepsilon)}}$	$\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\sqrt{\frac{2}{\varepsilon}+1}$
$\frac{\dot{U}(x,y,f)}{\omega_{\alpha\beta}W(f)}$	$\frac{1}{1+\varepsilon}\sqrt{\frac{2+\varepsilon}{2}}$	$\frac{1}{4(2+\varepsilon)}\sqrt{\frac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}}$	$\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\sqrt{\frac{2}{\varepsilon}-\frac{1}{1+\varepsilon}}$
$\frac{\ddot{U}(x,y,f)}{\omega_{\alpha\beta}^2 W(f)}$	$\sqrt{\frac{1}{1+\varepsilon}}$	$\frac{1}{2}\sqrt{\frac{3\varepsilon}{2+\varepsilon}}$	$\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\sqrt{\frac{2}{\varepsilon}-\frac{2}{1+\varepsilon}}$

where $E[U^2]$ is the ensemble mean of U^2 and S_F is the spectral density of the excitation. The optimum tuning frequency ratio and damping ratio for H_2 optimization of the system can be derived based on the fixed-points theory as [5]

$$T_d = \frac{1}{(1+\varepsilon)}\sqrt{1+\frac{2}{\varepsilon}} \tag{24a}$$

and

$$\zeta_{H_2} = \frac{1}{2}\sqrt{\frac{\varepsilon(4+3\varepsilon)}{2(1+\varepsilon)(2+\varepsilon)}}. \tag{24b}$$

If the forcing function $w(t)$ has power spectral density $S_F(f)$, the spectral density of the vibration response of the point (x, y) on the plate may be written as

$$S_u(x,y,f) = \left|\frac{U(x,y,f)}{W(f)}\right|^2 S_F(f). \tag{25}$$

With the optimum frequency and damping ratios as expressed in Eqs. (24a) and (24b), the mean square motion at point (x, y) can be derived as [5]

$$\sigma_u^2(x,y) = \frac{\omega_{\alpha\beta}}{2\pi}\int_{-\infty}^{\infty}\left|\frac{U(x,y,f)}{W(f)}\right|^2 S_F(f)\,\mathrm{d}f = \frac{\omega_{\alpha\beta}}{2}\left(\frac{a_{\alpha\beta}\varphi_{\alpha\beta}(x,y)}{\rho h\omega_{\alpha\beta}^2}\right)^2\sqrt{\frac{3\varepsilon+4}{\varepsilon(\varepsilon+1)}}. \tag{26}$$

2.2. *Optimization for minimizing the root mean square motion over the whole domain of the plate*

Using Eq. (17) and integrating the square of amplitude response over the whole domain of the plate, we may write

$$\int_0^a\int_0^b\left(\frac{U(x,y,f)}{W(f)}\right)^2\mathrm{d}y\,\mathrm{d}x = \int_0^{L_x}\int_0^{L_y}\left(\frac{1}{\rho h\omega_{\alpha\beta}^2}\right)^2\left(\sum_{p=1,q=1}^{\infty}\frac{\left(a_{pq}-\dfrac{b_{pq}\mu L_xL_y\sum_{p=1,q=1}^{\infty}\dfrac{a_{pq}\varphi_{pq}(x_o,y_o)}{\gamma_{pq}^2-f^2}}{\left(\dfrac{f^2-2\mathrm{j}\zeta Tf-T}{f^2(2\mathrm{j}\zeta Tf+T^2)}+\mu L_xL_y\sum_{p=1,q=1}^{\infty}\dfrac{b_{pq}\varphi_{pq}(x_o,y_o)}{\gamma_{pq}^2-f^2}\right)}\right)}{\gamma_{pq}^2-f^2}\varphi_{pq}(x,y)\right)^2\mathrm{d}y\,\mathrm{d}x. \tag{27}$$

Considering the orthogonality relations of the eigenfunctions, we may write

$$\int_0^{L_x}\int_0^{L_y}\varphi_{pq}(x,y)\varphi_{\alpha\beta}(x,y)\,\mathrm{d}y\,\mathrm{d}x = 0 \quad \text{if } p\neq\alpha \quad \text{or} \quad q\neq\beta$$

and

$$\int_0^{L_x}\int_0^{L_y}\varphi_{pq}(x,y)\varphi_{\alpha\beta}(x,y)\,\mathrm{d}y\,\mathrm{d}x = L_xL_y \quad \text{if } p=\alpha \quad \text{and} \quad q=\beta. \tag{28}$$

Eq. (27) can be simplified with the above orthogonality relations of the eigenfunctions as

$$\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x=\sum_{p=1,q=1}^{\infty}\left(\frac{L_xL_y}{\rho h\omega_{\alpha\beta}^2}\frac{\left(a_{pq}-\dfrac{b_{pq}\mu L_xL_y\sum_{p=1,q=1}^{\infty}\dfrac{a_{pq}\varphi_{pq}(x_o,y_o)}{\gamma_{pq}^2-f^2}}{\left(\dfrac{f^2-2\mathrm{j}\zeta Tf-T}{f^2(2\mathrm{j}\zeta Tf+T^2)}+\mu L_xL_y\sum_{p=1,q=1}^{\infty}\dfrac{b_{pq}\varphi_{pq}(x_o,y_o)}{\gamma_{pq}^2-f^2}\right)}\right)}{\gamma_{pq}^2-f^2}\right)^2. \tag{29}$$

For a structure with well-separated natural frequencies, the mean square modal displacement response in the vicinity of the $\alpha\beta$th natural frequency may be approximated by considering $p=\alpha$ and $q=\beta$ and ignoring other modes. Eq. (29) may be written as

$$\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x=\left(\frac{L_xL_y}{\rho h\omega_{\alpha\beta}^2}\frac{\left(a_{\alpha\beta}-b_{\alpha\beta}\dfrac{\mu L_xL_y\dfrac{a_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-f^2}}{\left(\dfrac{f^2-2\mathrm{j}\zeta Tf-T}{f^2(2\mathrm{j}\zeta Tf+T^2)}+\mu L_xL_y\dfrac{b_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-f^2}\right)}\right)}{1-f^2}\right)^2. \tag{30}$$

The root mean square amplitude response of the vibrating plate with an absorber may be written as

$$\begin{aligned}\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}&=\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\omega_{\alpha\beta}^2}\right)\left(\frac{1-b_{\alpha\beta}\dfrac{\mu L_xL_y\dfrac{\varphi_{\alpha\beta}(x_o,y_o)}{1-f^2}}{\left(\dfrac{f^2-2\mathrm{j}\zeta Tf-T}{f^2(2\mathrm{j}\zeta Tf+T^2)}+\mu L_xL_y\dfrac{b_{\alpha\beta}\varphi_{\alpha\beta}(x_o,y_o)}{1-f^2}\right)}}{1-f^2}\right)\\&=\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\omega_{\alpha\beta}^2}\right)\left[\frac{T^2-f^2+2\mathrm{j}\zeta Tf}{(2\mathrm{j}\zeta Tf+T^2-f^2)(1-f^2)-\varepsilon f^2(2\mathrm{j}\zeta Tf+T^2)}\right]=\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\omega_{\alpha\beta}^2}\right)Z(f).\end{aligned} \tag{31}$$

The magnitude of the root-mean-square amplitude response of the plate may be written as

$$\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right|=\left|\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right||Z(f)|. \tag{32}$$

The H_∞ optimizations for minimizing root mean square motion, velocity and acceleration responses of the whole plate are written, respectively, as

$$H_{\infty\_\mathrm{plate\_disp}}=\inf_f\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right|\right)\right), \tag{33a}$$

$$H_{\infty\_\text{plate\_vel}} = \inf_f\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{\dot{U}(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right|\right)\right) = \inf_f\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{\mathrm{j}fU(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right|\right)\right) \tag{33b}$$

and

$$H_{\infty\_\text{plate\_acc}} = \inf_f\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{\ddot{U}(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right|\right)\right) = \inf_f\left(\sup_{\zeta,T}\left(\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{-f^2U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right|\right)\right). \tag{33c}$$

The optimum tuning frequency and damping ratios and the height of the fixed points in the frequency spectrum of the primary system in Eqs. (33a)–(33c) for H_∞ optimization can be derived based on the fixed-points theory as in the case of the sdof system [5,6] and the results are listed in Table 2.

The objective of H_2 optimization in this case is to minimize the vibration energy of the whole plate of all frequencies of the system. The performance index in this case may be defined as

$$H_{2\_\text{plate}} = \inf_{\zeta,T}\left(\frac{E\left[\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x\right]}{2\pi S_f\omega_{\alpha\beta}/D^2}\right). \tag{34}$$

The frequency and damping ratios for H_2 optimization of the system can be derived based on the fixed-points theory as [5]

$$T_{H_2} = \sqrt{\frac{\varepsilon+2}{\varepsilon(1+\varepsilon)^2}} \tag{35a}$$

and

$$\zeta_{H_2} = \frac{1}{2}\sqrt{\frac{\varepsilon(4+3\varepsilon)}{2(1+\varepsilon)(2+\varepsilon)}}. \tag{35b}$$

The H_2 optimization is the minimization of the root mean square motion response over the whole domain of the plate under wide-band random excitation. With optimum frequency and damping ratios as

Table 2
The approximated H_∞ tuning of the plate for control of vibration of the whole plate and the height of the fixed points in the response spectrum

Transfer function	Tuning ratio	Damping ratio	Height of the fixed points in the response spectrum
$\left\|\sqrt{\int_0^a\int_0^b\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right\|$	$\frac{1}{1+\varepsilon}$	$\sqrt{\frac{3\varepsilon}{8(1+\varepsilon)}}$	$\left(\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right)^2\left(\frac{2}{\varepsilon}+1\right)$
$\left\|\sqrt{\int_0^a\int_0^b\left(\frac{\dot{U}(x,y,f)}{\omega_{\alpha\beta}W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right\|$	$\frac{1}{1+\varepsilon}\sqrt{\frac{2+\varepsilon}{2}}$	$\frac{1}{4(2+\varepsilon)}\sqrt{\frac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}}$	$\left(\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right)^2\left(\frac{2}{\varepsilon}-\frac{1}{1+\varepsilon}\right)$
$\left\|\sqrt{\int_0^a\int_0^b\left(\frac{\ddot{U}(x,y,f)}{\omega_{\alpha\beta}^2W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right\|$	$\sqrt{\frac{1}{1+\varepsilon}}$	$\frac{1}{2}\sqrt{\frac{3\varepsilon}{2+\varepsilon}}$	$\left(\frac{a_{\alpha\beta}}{\rho h\omega_{\alpha\beta}^2}\right)^2\left(\frac{2}{\varepsilon}-\frac{2}{1+\varepsilon}\right)$

expressed in Eqs. (35a) and (35b), the total mean square motion of the whole plate can be derived as [5]

$$\frac{\omega_{\alpha\beta}}{2\pi}\int_{-\infty}^{\infty}\left|\sqrt{\int_0^{L_x}\int_0^{L_y}\left(\frac{U(x,y,f)}{W(f)}\right)^2 \mathrm{d}y\,\mathrm{d}x}\right|^2 S_w(f)\,\mathrm{d}f = \frac{\omega_{\alpha\beta}}{2}\left(\frac{a_{\alpha\beta}L_xL_y}{\rho h\omega_{\alpha\beta}^2}\right)^2\sqrt{\frac{3\varepsilon+4}{\varepsilon(\varepsilon+1)}}. \quad (36)$$

3. Simulation results and discussion

3.1. Case 1: Minimization of the mean square vibration at one point of a plate (H_2 optimization)

To test the usefulness of the derived H_2 optimization solution for suppressing vibrations in plates, the numerical case studied by Jacquot [9] was analyzed with the optimum tuning derived in the previous section and the results were compared to those obtained by Jacquot. The vibration of a square plate with four sides simply supported was considered. The eigenfunctions may be written as

$$\varphi_{pq} = 2\,\sin(p\pi x)\sin(q\pi y). \quad (37)$$

The excitation was stationary and random in time, i.e. $g(x,y)=1$, and it was uniformly applied on the plate. In this case,

$$a_{pq} = \frac{8}{pq\pi^2}, \quad p,q = 2n-1, \quad n \in N$$
$$\text{else} \quad a_{pq} = 0, \quad (38)$$

$$b_{pq} = \varphi_{pq}(x_o,y_o) = 2\sin p\pi x_o\,\sin q\pi y_o, \quad p,q \in N. \quad (39)$$

The dimensions of the plate were $a=1$ m, $b=1$ m and $h=0.01$ m. The material of the plate was aluminum of $\rho = 2.71\times10^3\,\mathrm{kg\,m^{-3}}$, $E = 6.9\times10^9$ Pa and $v = 0.33$. In the analysis made by Jacquot [9], the frequency ratio was chosen as 1. The vibration mode required to be suppressed was $\alpha=\beta=1$. The attachment position of the absorber on the plate was $x_o = y_o = 0.5$. The mass ratio and damping ratio for minimum mean square motion at the attachment point were found to be 0.275 and 0.45, respectively, by Jacquot. In the current analysis, the same mass ratio was used so that the result of vibration suppression could be compared to that of Jacquot. The modal response amplitude at the point of attachment $\varphi_{11}(x_o,y_o)$ was 2 and therefore ε was 1.1 according to Eq. (21b). The optimum frequency and damping ratios in this case were calculated to be 0.5929 and 0.3927, respectively, in applying Eqs. (24a) and (24b). The vibration amplitude response at point (x_o,y_o) of the plate was calculated according to Eq. (17). The spectral density of the vibration amplitude response at point (x_o,y_o) was calculated according to Eq. (25) and it was plotted in Fig. 2 and compared with the corresponding curve by Jacquot (Eqs. (25) and (28) of Ref. [9]). The spectral density of the vibration amplitude response at point (x_o,y_o) for the case of no absorber added was also plotted for comparison. It could be observed in Fig. 2 that both Jacquot's result and the present result provided vibration control at point (x_o,y_o) of the plate. However, the mean square motion at point (x_o,y_o) of the plate with the proposed frequency and damping ratios was found to be 55.8% smaller than that obtained by Jacquot. Jacquot also reported that there was an optimum mass ratio leading to minimum mean square motion of the plate but no particular optimum mass ratio could be found in applying the present theory. Based on Eqs. (21b) and (26), it was observed that the mass ratio should be as high as possible in order to reduce the mean square motion of the plate.

The exact values of tuning frequency and damping ratios for minimum mean square motion at the attachment point of the plate were determined numerically with Eq. (17) as $T = 0.5854$ and $\zeta = 0.4162$. The difference of mean square motion of the plate at point (x_o,y_o) using the proposed and the exact set of T and ζ was found to be 0.14%. This shows that the proposed optimum tuning frequency and damping ratios are quite accurate even though they are determined based on the vibration response of only one mode of the plate.

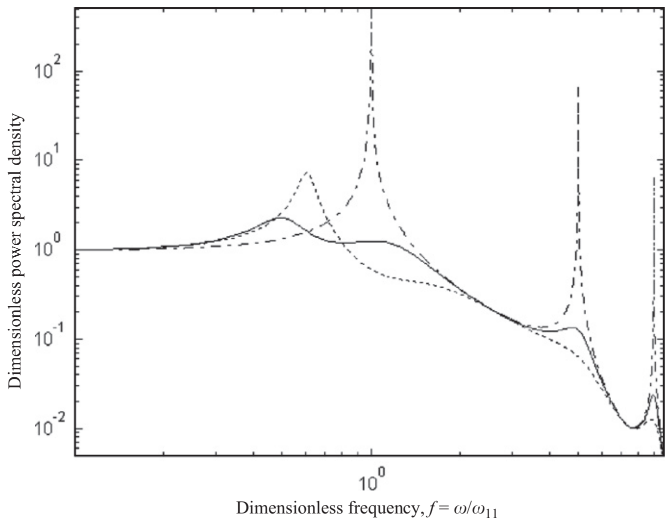

Fig. 2. Dimensionless motion power spectral density of a square plate with $g(x, y) = 1$, $\mu = 0.275$, $x_o = y_o = a/2$. -------, Jacquot's result [9]; ——, present theory (Eq. (25)); ----, no absorber added.

3.2. Case 2: Minimization of the kinetic energy of a vibrating beam (H_∞ optimization)

To test the usefulness of the derived H_∞ optimization solution for suppressing vibrations, a continuous vibrating system, the numerical testing case for the minimization of the maximum kinetic energy of a vibrating beam reported by Dayou [10], was studied by applying the present theory and the result was compared with those obtained by Dayou. The vibrating beam considered by Dayou was a simply supported aluminum beam excited by a point force of unit amplitude at $0.1L$ as shown in Fig. 3. The eigenfunctions and eigenvalues of the beam could be written, respectively, as [13]

$$\varphi_p(x) = \sin\left(\frac{p\pi x}{L}\right), \quad p = 1, 2, 3, \ldots \tag{40}$$

and

$$\omega_p^2 = \left(\frac{p\pi}{L}\right)^4 \left(\frac{EI}{\rho A}\right), \quad p = 1, 2, 3 \ldots \tag{41}$$

where $L = 1\,\text{m}$, $E = 207\,\text{GPa}$, $I = 8.1295 \times 10^{-10}\,\text{m}^4$, $\rho = 7870\,\text{kg/m}^3$ and $A = 2.42 \times 10^{-4}\,\text{m}^2$. A DVA was attached at $x_o = 0.5L$ and mass ratio, μ, was 0.05.

From Eqs. (7) and (8), we have

$$a_p = \frac{\int_0^L \delta(x - x_1) \sin\left(\frac{p\pi x}{L}\right) \mathrm{d}x}{\int_0^L \sin^2\left(\frac{p\pi x}{L}\right) \mathrm{d}x} = \frac{2}{L} \sin\left(\frac{p\pi x_1}{L}\right) \tag{42a}$$

and

$$b_p = \frac{\sin\left(\frac{p\pi x_o}{L}\right)}{\int_0^L \sin^2\left(\frac{p\pi x}{L}\right) \mathrm{d}x} = \frac{2}{L} \sin\left(\frac{p\pi x_o}{L}\right), \quad p = 1, 2, 3, \ldots . \tag{42b}$$

The optimization problem could be expressed as

$$H_{\infty\_\text{beam\_vel}} = \inf_f \left(\sup_{\zeta, T} \left(\left| \int_0^L \left(\frac{\dot{U}(x, f)}{\omega_1 W(f)} \right)^2 \mathrm{d}x \right| \right) \right) = \inf_f \left(\sup_{\zeta, T} \left(\left| \int_0^L \left(\frac{\mathrm{j} f U(x, f)}{\omega_1 W(f)} \right)^2 \mathrm{d}x \right| \right) \right), \tag{43}$$

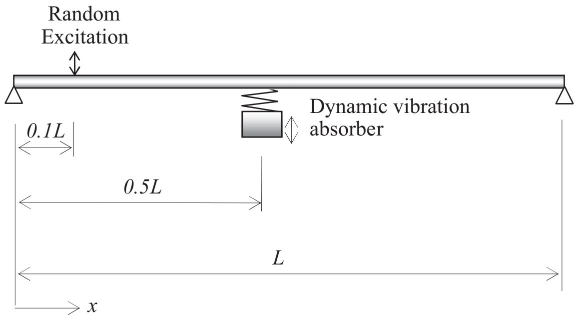

Fig. 3. Schematics of a simply supported beam with a vibration absorber excited by a random force at $x = 0.1L$.

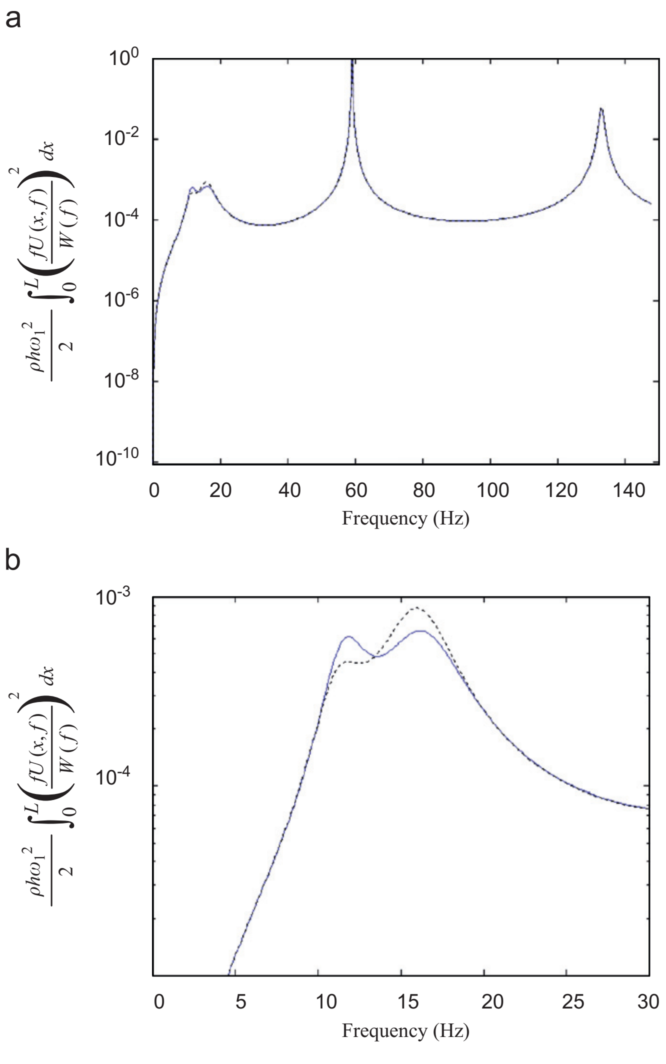

Fig. 4. Kinetic energy (in J/N^2) of a simply supported beam with optimum vibration absorber fitted at $x_o = 0.5L$ with the first natural frequency as the control target: (a) figure showing all three modes; (b) in the vicinity of the first mode. - - - - - -, $T = 0.8333$; $\zeta = 0.25$ [10]; ——, $T = 0.8775$, $\zeta = 0.2556$ (present theory).

where

$$\frac{U(x,f)}{W(f)} = \left(\frac{a_1 L}{\rho h \omega_1^2}\right)\left[\frac{T^2 - f^2 + 2\mathrm{j}\zeta Tf}{(2\mathrm{j}\zeta Tf + T^2 - f^2)(1 - f^2) - \varepsilon f^2(2\mathrm{j}\zeta Tf + T^2)}\right] \tag{44}$$

and $\varepsilon = \mu\varphi_1^2(x_o)$, which was the one-dimensional version of the ε used in the theory section.

According to Dayou [10], the optimum frequency and damping ratios were $1/1+\varepsilon = 0.8333$ and $\sqrt{3\varepsilon/8(1+\varepsilon)} = 0.25$, respectively. Based on the present theory and the derived expressions of the optimum frequency and damping ratios for H_∞ optimization with different types of transfer functions as shown in Table 1, the frequency and damping ratios for minimum kinetic energy amplitude of the plate were

$$T = \frac{1}{1+\varepsilon}\sqrt{\frac{2+\varepsilon}{2}} = 0.8740 \quad \text{and} \quad \zeta = \frac{1}{4(2+\varepsilon)}\sqrt{\frac{\varepsilon(24+24\varepsilon+5\varepsilon^2)}{1+\varepsilon}} = 0.2498, \text{ respectively.}$$

Kinetic energy amplitudes of the whole beam at steady state were calculated at different excitation frequencies according to the following equation:

$$\frac{\rho h \omega_1^2}{2}\int_0^L \left[\frac{fU(x,f)}{W(f)}\right]^2 \mathrm{d}x = \frac{\rho h \omega_1^2}{2}\sum_{p=1}^{p_{\max}}\left(\frac{Lf}{\rho h \omega_1^2}\right)^2 \frac{\left(a_p - \dfrac{b_p \mu L \sum_{p=1}^{p_{\max}} \dfrac{a_p \varphi_p(x_o)}{\gamma_p^2 - f^2}}{\left(\dfrac{f^2 - 2\mathrm{j}\zeta Tf - T}{f^2(2\mathrm{j}\zeta Tf + T^2)} + \mu L \sum_{p=1}^{p_{\max}} \dfrac{b_p \varphi_p(x_o, y_o)}{\gamma_p^2 - f^2}\right)}\right)^2}{(\gamma_p^2 - f^2)^2}. \tag{45}$$

A Matlab program was written to calculate these kinetic energy amplitudes and the results are plotted in Fig. 4a. Ten vibration modes ($p_{\max} = 10$) of the beam were used in the calculation. Both the kinetic energy of the beam calculated based on the present theory and that by Dayou are plotted in Fig. 4a for comparison. The amplitude of the kinetic energy at the first resonance of the beam was suppressed after adding the vibration absorber. However, Fig. 4b shows a close-up of the spectrum around the first natural frequency of the beam, and the heights of the two peaks of the curve of Dayou had a big difference, indicating that the damping and frequency ratios of the absorber were not optimal based on the fixed-points theory [3]. The maximum amplitude of the kinetic energy of the whole beam around the first natural frequency of the beam calculated with the proposed frequency and damping ratios was found to be about 32% smaller than that of the beam with the frequency ratio ($T = 0.8333$) and damping ratio ($\zeta = 0.25$) used by Dayou [10].

The exact values of tuning frequency and damping ratios to minimize the maximum amplitude of the kinetic energy of the whole beam around the first natural frequency of the beam were determined numerically with Eq. (45) as $T = 0.8775$ and $\zeta = 0.2556$. The difference of this maximum amplitude of kinetic energy in using the proposed and the exact sets of T and ζ was found to be about 3%.

4. Conclusion

In this article, we have derived analytical solutions to the H_∞ and H_2 optimization problems of DVA attached to a vibrating plate under random excitation. Expressions of the optimum tuning frequency and damping ratios are derived for the absorber assuming single-mode vibration of the plate.

The optimum tuning frequency and damping ratios of the absorber derived in the present theory for solving the H_∞ and H_2 optimization problems applied to vibrating plate structures have similar forms to those of the sdof system. However, the tuning equations are based on the equivalent mass ratio ε, which is a function of both the mass ratio and the position of the absorber on the plate structure. Moreover, it is derived that both the optimum tuning frequency and the damping ratios for minimum vibration at a certain point are the same as those in the case of the minimum mean square motion for the whole plate. That means the mean square motion would be minimum when the vibration at a single point of the surface is minimum.

Secondly, the vibration response in H_∞ optimization and the mean square motion in H_2 optimization would be reduced when the equivalent mass ratio ε is increased under the optimum tuning condition.

That means a higher mass ratio and an attachment point of the absorber having higher modal response should be chosen for the suppression of vibration for the whole plate or at one point of the plate. This finding is different from that of Jacquot [10] who showed that there would be an optimum mass ratio for minimum mean square motion of a vibrating plate under random excitation.

Thirdly, based on the expressions as shown in Tables 1 and 2 for the heights of the fixed points in the response spectrum for H_∞ optimization, it is found that the heights of the fixed points in the (dimensionless) displacement response spectrum are higher than those of the (dimensionless) velocity response spectrum and in turn higher than those of the (dimensionless) acceleration response spectrum.

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